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Mechanics and ContinuumThermodynamics

A brief account of the mathematical modelling

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Summary The Mechanics and Thermodynamics of chemically stable, continuous bodies can be built up from the intuitive notion, that a theory for the computational simulation of the development in time of observable phenomena needs at least one quantity, that remains constant in time, in order to have a check on the validity of the theory for all admissible initial data. The central role, that can be given to the principle of conservation of energy, is demonstrated, both for the Mechanics of rigid bodies as well as for the Mechanics and Thermodynamics of gases and deformable bodies.

Key words inertial system, rigid bodies, state space, local equilibrium, physical space

1
Introduction

In the Natural Sciences we have a useful theory, if it enables us to simulate the development in time of observable phenomena by computation, be it with a limited, but known degree of accuracy. And simulation is something else than prediction, as we know from Chaos Theory.

In the Galilëian thermo-mechanics the mathematical description of phenomena uses Euclidean Geometry with time as a parameter, that varies continuously over the range of real numbers.

A concise formulation of the theory is obtained with the aid of Linear Algebra, Vector- and Tensorcalculus. The computations have of course to be carried out with vector components and tensor components in a suitably chosen coordinate system.

For computational purposes the mathematical modelling must ultimately result in a finite dimensional system of equations.

For rigid bodies these equations are ordinary, second order differential equations, to be replenished by ordinary, first order differential equations for models of rigid bodies, for which the orientation of the so-called principal axes has to be traced. Proper initial condities are the initial position and orientation, as well as the initial velocities of the rigid bodies.

For deformable bodies we derive for so-called continuum models systems of partial differential equations for the field quantities of these models. Second order equations for the displacements and equations for the velocities, that determine the heatflow, have to be replenished by the constitutive equations for the so-called state variables. Generally these equations can only be given in the form of rate equations, since the irreversibility of the processes precludes a unique relation between the state at an arbitrary instant of time and the initial state. Initial conditions are initial displacement and velocity fields, and a full specification of the initial fields of state variables. Furthermore the proper boundary conditions have to be supplied.

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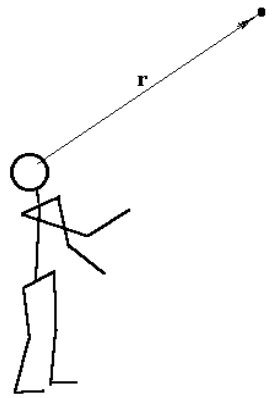
In the case of a deformable body a straightforward discretization procedure consists of a splitting up the body into an assemblage of small finite elements, interconnected in nodal points such that the continuity of the displacement and velocity fields at the common boundaries is at all times ensured. The fields of state variables are represented by finite dimensional representations of these variables in what can be called sampling (or integration) points inside the finite elements. The rate equations for the elements of these vectors of state quantities are adopted directly from the continuum model for each finite element. The equations for the nodal quantities in the finite element model are derived by applying the energy equation in the same way as it can be applied to derive the equations of continuum thermodynamics, by what has been called the principles of virtual power and of virtual heat [1].
In the case of large inelastic deformations of a body, and for problems of flow of gases and liquids, the finite element method in terms of material elements is not applicable and the space instead of the body has to be divided into finite elements. For the details we refer to the specialized literature.

2
Theory for one point mass



It is always possible to position the observer such, that at any time holds

$$\mathrm{d}^2/\mathrm{dt}^2\, \boldsymbol{r} \equiv \ddot{\boldsymbol{r}} = \boldsymbol{0} \quad \text{or} \quad \mathrm{d}/\mathrm{dt}\, \boldsymbol{r} \equiv \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_0. \tag{1}$$

According to linear algebra a scalar product of $\ddot{\boldsymbol{r}}$ and $\dot{\boldsymbol{r}}$, that is equal to zero for all $\dot{\boldsymbol{r}}$,

$$\langle \ddot{\boldsymbol{r}}, \dot{\boldsymbol{r}} \rangle = 0 \quad \forall \dot{\boldsymbol{r}},$$

implies $\ddot{\boldsymbol{r}} = \boldsymbol{0}$.
Since the theory must hold for an arbitrary choice of the initial time and of the initial velocity $\dot{\boldsymbol{r}}_0$, the theory is couched in the statement that the inner product of the velocity vector is constant in time:

$$\dot{\boldsymbol{r}} \circ \dot{\boldsymbol{r}} = \text{constant}. \tag{2}$$

From

$$\frac{\mathrm{d}}{\mathrm{dt}}(\dot{\boldsymbol{r}} \circ \dot{\boldsymbol{r}}) = 0 \quad \forall \dot{\boldsymbol{r}}$$

we have again the result

$$\langle \ddot{\boldsymbol{r}}, \dot{\boldsymbol{r}} \rangle = 0 \ \forall \dot{\boldsymbol{r}} \Rightarrow \ddot{\boldsymbol{r}} = \boldsymbol{0}. \tag{3}$$

However the theory has no physical content and is for the observer only descriptive.

3
Theory for two point masses

Suppose an observer can be positioned such that for two non-interacting point masses A and B holds for all times

$$\dot{\boldsymbol{r}}_\mathrm{A} \circ \dot{\boldsymbol{r}}_\mathrm{A} = \text{ constant and } \dot{\boldsymbol{r}}_\mathrm{B} \circ \dot{\boldsymbol{r}}_\mathrm{B} = \text{ constant},$$

then the observer concludes:

$$\ddot{\boldsymbol{r}}_\mathrm{A} = \boldsymbol{0} \text{ and } \ddot{\boldsymbol{r}}_\mathrm{B} = \boldsymbol{0}.$$

The observer is positioned in a so-called **inertial system.** Inertial systems are determined but for a constant rotation and a constant velocity with respect to each

other (**Galileï**). Note that we postulate that the initial velocities of all point masses in an inertial system are arbitrary, independent of each other.
The theory becomes more interesting if we suppose the two point masses to be interacting. Again we shall derive the theory from the hypothesis that there is one scalar quantity E, that remains constant in time. In this quantity the inner products of the velocity vectors of the masses A and B will be represented, but intuitively we feel that these inner products will at least have to be multiplied by weight factors $\mathrm{m_A}$ and $\mathrm{m_B}$. It is to be expected that that the mutual interaction will be weaker the farther away from each other these point masses are. We therefore propose that the mutual interaction will be inversely proportional to the distance and proportional to each of the weight factors. Hence we add to the sum of the inner product terms in E a term with the product of the weight factors in the numerator and the relative distance in the denominator. Since this term must have the same physical dimension as the inner product terms, the extra term has to be multiplied by a physical constant of the proper physical dimension. This constant we shall denote by G:

$$\mathrm{E} = \frac{1}{2}\mathrm{m_A}(\dot{\boldsymbol{r}}_\mathrm{A} \circ \dot{\boldsymbol{r}}_\mathrm{A}) + \frac{1}{2}\mathrm{m_B}(\dot{\boldsymbol{r}}_\mathrm{B} \circ \dot{\boldsymbol{r}}_\mathrm{B}) - \frac{\mathrm{G m_A m_B}}{\left\{(\boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B}) \circ (\boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B})\right\}^{1/2}} \tag{4}$$

E = constant in time for arbitrary initial velocities implies

$$\dot{\mathrm{E}} = \langle \mathrm{m_A}\ddot{\boldsymbol{r}}_\mathrm{A} - \boldsymbol{f}_\mathrm{AB}, \dot{\boldsymbol{r}}_\mathrm{A}\rangle + \langle \mathrm{m_B}\ddot{\boldsymbol{r}}_\mathrm{B} - \boldsymbol{f}_\mathrm{BA}, \dot{\boldsymbol{r}}_\mathrm{B}\rangle = 0 \quad \forall \dot{\boldsymbol{r}}_\mathrm{A}, \dot{\boldsymbol{r}}_\mathrm{B}\ ,$$

or

$$\mathrm{m_A}\ddot{\boldsymbol{r}}_\mathrm{A} = \boldsymbol{f}_\mathrm{AB} \quad \text{and} \quad \mathrm{m_B}\ddot{\boldsymbol{r}}_\mathrm{B} = \boldsymbol{f}_\mathrm{BA}\ , \tag{5}$$

where

$$\boldsymbol{f}_\mathrm{AB} = -\boldsymbol{f}_\mathrm{BA} = -\frac{\mathrm{G m_A m_B}(\boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B})}{\left\{(\boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B}) \circ (\boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B})\right\}^{3/2}}\ . \tag{6}$$

The vectors $\boldsymbol{f}_\mathrm{AB}$ and $\boldsymbol{f}_\mathrm{BA}$ are what **Newton** called **forces**, equal to **mass** multiplied by **acceleration** according to (5), and determined by the **law of gravity** with (6).
The equations (5) and (6) are in accordance with the observation that everywhere on earth the acceleration due to gravity is the same for all objects (apart from air resistance, etc.).
We have introduced three physical dimensions. The length of the shortest distance between two points is by a multiplying factor expressed in terms of meters, based upon a standard *m.* The time interval between two observations is expressed in terms of seconds, based upon a standard *s*. Finally the mass of an object is expressed in terms of kilograms, the multiplying factor based upon a standard *kg*. A platinum bar, stored at constant temperature, may serve as the standard *m*, a weight, stored in an non-aggressive environment represents the standard *kg,* while the standard *s* is derived from observable phenomena, that repeat themselves regularly in time. Initially the standard *s* was derived from the motion of the earth with respect to the sun. Now the standards *m* and *s* are connected with atomic phenomena, because they can be defined with much greater precision that way.
Cavendish devised an experiment for the measurement of the force of mutual attraction of two masses (1797-1798). This experiment was repeated with increasing accuracy by Boys (1889-1895) and by Hey and Chrzanowski (1942). These experiments gave for G a value of $(6.673 \pm 0.003)10^{-11} kg^{-1}m^3s^{-2}$.

As in the next paragraph will be explained the motion of the celestial bodies in our solar system can be simulated with great accuracy by extending the theory for two point masses, set forth above, to a theory for *n* point masses. For the determination of the motions of their so-called centres of gravity the celestial bodies, with their finite dimensions, appear to be representable by masses, concentrated in mathematical points.

4

Rigid bodies of finite dimensions

In the **continuum theory** the mass of bodies of finite dimensions is taken into account by a continuous distribution, represented by a **massdensity** ρ (the physical dimension is mass per unit volume). In general this massdensity will be a function of the place in the body, i.e. a function of the position vector from one and for all places the same material point of the body. These position vectors we shall denote by $\boldsymbol{r}^*$ and their common origin may be determined such that it will be called the **centre of gravity** of the body, for which holds

$$\int_B \rho\, \boldsymbol{r}^* \mathrm{d}V = 0\,. \tag{7}$$

A vector $\boldsymbol{r}^*$, rigidly connected to the body, has a constant length if the body is not deformable, but its orientation with respect to the inertial system of the observer will be changing with the motion of the body. If this vector at an initial time is denoted by $\boldsymbol{r}_0^*$, the vector $\boldsymbol{r}^*$ at a later time can be expressed in terms of $\boldsymbol{r}_0^*$ by a linear, so-called **orthogonal transformation** $\boldsymbol{R}$.

$$\boldsymbol{r}^* = \boldsymbol{R}\boldsymbol{r}_0^*\,,\ \boldsymbol{r}^* \circ \boldsymbol{r}^* = \boldsymbol{r}_0^* \circ (\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{r}_0^*) = \boldsymbol{r}_0^* \circ \boldsymbol{r}_0^* \Rightarrow \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}\,. \tag{8}$$

The vector $\boldsymbol{r}^*$ changes with time in accordance with the time dependent transformation $\boldsymbol{R}$, the transpose of which is indicated by $\boldsymbol{R}^{\mathrm{T}}$. Since the product of $\boldsymbol{R}$ with its transpose equals the unit transformation $\boldsymbol{I}$, the time derivative of this product equals zero and consequently we have

$$\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} = -\boldsymbol{R}\dot{\boldsymbol{R}}^{\mathrm{T}} = \boldsymbol{\Omega}, \quad \boldsymbol{\Omega} = -\boldsymbol{\Omega}^{\mathrm{T}}. \tag{9}$$

Through this skew symmetric transformation $\boldsymbol{\Omega}$ the time derivative of the vector $\boldsymbol{r}^*$ can be expressed in terms of the axial vector $\boldsymbol{\omega}$, representing the **angular velocity** about an axis of rotation in rad/s:

$$\dot{\boldsymbol{r}}^* = \dot{\boldsymbol{R}}\boldsymbol{r}_0^* = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{r}_0^* = \boldsymbol{\Omega}\boldsymbol{r}^* = \boldsymbol{\omega} \wedge \boldsymbol{r}^*\,. \tag{10}$$

If we denote the position vector of the centre of gravity of the body in the inertial system by $\boldsymbol{r}^{\mathrm{c}}$, then the position vector, the velocity vector and the acceleration vector of a material point of the body are given by:

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}^{\mathrm{c}} + \boldsymbol{r}^*, \\ \dot{\boldsymbol{r}} &= \dot{\boldsymbol{r}}^{\mathrm{c}} + \boldsymbol{\omega} \wedge \boldsymbol{r}^*, \\ \ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}^{\mathrm{c}} + \dot{\boldsymbol{\omega}} \wedge \boldsymbol{r}^* + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{r}^*). \end{aligned} \tag{11}$$

The third term in the acceleration is called the centripetal acceleration. Since in this term also appears the angular velocity $\boldsymbol{\omega}$, together with the velocity, that in the derivation of the equations of motion is to be treated as an arbitrarily to be chosen initial velocity, we shall henceforth express the arbitrarily to be chosen velocity in terms of the so-called **virtual** velocity of the centre of gravity, $\delta\dot{\boldsymbol{r}}^{\mathrm{c}}$, and of the **virtual** angular velocity $\delta\,\boldsymbol{\omega}$ about an axis through the centre of gravity. Thus for

$$\frac{d}{dt}\int_B \tfrac{1}{2}\rho\ \dot{\boldsymbol{r}}\circ\dot{\boldsymbol{r}}dV = \int_B \rho\ \langle\ddot{\boldsymbol{r}},\dot{\boldsymbol{r}}\rangle dV\ \ \forall\dot{\boldsymbol{r}}$$

we write

$$\frac{d}{dt}\int_B \tfrac{1}{2}\rho\ \dot{\boldsymbol{r}}\circ\dot{\boldsymbol{r}}dV =$$

$$\int_B \langle \frac{d}{dt}\left\{\rho(\dot{\boldsymbol{r}}^c + \boldsymbol{\omega}\wedge\boldsymbol{r}^*)\ \right\},(\delta\dot{\boldsymbol{r}}^c + \delta\ \boldsymbol{\omega}\wedge\boldsymbol{r}^*)dV\ \forall\delta\ \dot{\boldsymbol{r}}^c,\delta\ \boldsymbol{\omega} \qquad (12)$$

Making use of the properties of the scalar product, $\langle,\rangle$, of the vector product, $\wedge$, and of the tensor product, $\otimes$, and taking into account (7), we may write (12) in the more conveniently arranged form:

$$\langle\delta\dot{\boldsymbol{r}}^c,\frac{d}{dt}\left\{\dot{\boldsymbol{r}}^c\int_B \rho dV\ \right\}\rangle + \langle\delta\ \boldsymbol{\omega},\frac{d}{dt}\left[\ \int_B \rho\left\{(\boldsymbol{r}^*\circ\boldsymbol{r}^*)\boldsymbol{I} - \boldsymbol{r}^*\otimes\boldsymbol{r}\ \right\}dV\ \boldsymbol{\omega}\ \right]\rangle =$$

$$\langle\delta\dot{\boldsymbol{r}}^c, m_B\ddot{\boldsymbol{r}}^c\rangle + \langle\delta\ \boldsymbol{\omega},\frac{d}{dt}(\boldsymbol{J}\boldsymbol{\omega})\rangle. \qquad (13)$$

Here $\boldsymbol{I}$ is the unit tensor. The so-called inertia tensor $\boldsymbol{J}$ is a positive definite tensor and it can be expressed in terms of its three principal eigenvalues, J_1, J_2, J_3, and in the three corresponding orthonormal eigenvectors, $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$.

$$\boldsymbol{J} = \sum_{i=1}^{3} J_i(\boldsymbol{p}_i\otimes\boldsymbol{p}_i). \qquad (14)$$

The three principal eigenvalues are the so-called **principal moments of inertia.** Axes along the eigenvectors are denoted as the **principal axes of inertia**.

In case a body possesses axial symmetry one of the principal axes of inertia is the axis of symmetry, any axis perpendicular to this axis of symmetry is a principal axis and the other two principal moments of inertia are equal in value.

For a body with spherical symmetry all axes through the centre of gravity are principal axes and the three principal moments of inertia are equal in value.

If the body has an angular velocity $\boldsymbol{\omega}$ with respect to the inertial system, then the inertia tensor is time dependent, except in the case of a body with axial symmetry and an axis of rotation along the axis of symmetry, and except in the case of a body with spherical symmetry.

When we assume that in the interaction between a mass point A and a body B of finite dimensions the contributions of infinitesimal masses dm of the body B have the same form as the contribution of the mass point B in (4), then the second term in (4) is to be replaced by

$$-Gm_A\int_B \frac{dm}{\left\{(\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*)\circ(\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*)\right\}^{1/2}} \qquad (15)\ .$$

If the diameter of body B is small compared to the distance between A and B,

$$\boldsymbol{r}^*\circ(\boldsymbol{r}_A - \boldsymbol{r}_B^c) << (\boldsymbol{r}_A - \boldsymbol{r}_B^c)\circ(\boldsymbol{r}_A - \boldsymbol{r}_B^c),$$

then all mass of the body B may be concentrated in the centre of gravity without great loss of accuracy. Moreover in case of a body with spherical symmetry this may be done without any loss of accuracy as it will be shown in the following.

Using the notation

$$\Delta r = \left\{(\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*) \circ (\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*)\right\}^{1/2}, r^* = (\boldsymbol{r}^* \circ \boldsymbol{r}^*)^{1/2},$$

we have in spherical coordinates r^*, φ, ϑ

$$(\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*) \circ (\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*) = \Delta r^2 + r^{*2} - 2\Delta r r^* \cos\vartheta.$$

The expression for dm in the integral (15),

$$dm = \rho r^{*2} dr^* \sin\vartheta d\vartheta d\varphi,$$

contains the derivative with respect to ϑ of the denominator. Consequently we have

$$-Gm_A \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\rho r^{*2} dr^* \sin\vartheta d\vartheta d\varphi}{\left(\Delta r^2 + r^{*2} - 2\Delta r r^* \cos\vartheta\right)^{1/2}} =$$

$$-\frac{Gm_A}{\Delta r} 2\pi \int_0^R \rho r^* \left\{(\Delta r + r^*) - \left|\Delta r - r^*\right|\right\} dr^*.$$

For $\Delta r > r^*$ the integrand is the mass of a spherical shell of thickness $dr^*, dm = 4\rho\pi r^{*2} dr^*$. Hence without any loss of accuracy the mass of body B may be concentrated in its centre of gravity.

$$-Gm_A \int_B \frac{dm}{\left\{(\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*) \circ (\boldsymbol{r}_A - \boldsymbol{r}_B^c - \boldsymbol{r}^*)\right\}^{1/2}} = -\frac{Gm_A m_B}{\Delta r} \tag{16}$$

Since for bodies with spherical symmetry the interaction solely depends on the distances of their centres of gravity, the angular velocities about axes through these centres of gravity are independent of each other and may be considered for each body seperately. If for such bodies the moment of inertia is denoted by J $(J_1 = J_2 = J_3 = J)$, then for each body holds

$$\langle \boldsymbol{\delta\omega}, J\dot{\boldsymbol{\omega}} \rangle = 0 \ \forall \boldsymbol{\delta\omega} \Rightarrow \dot{\boldsymbol{\omega}} = \boldsymbol{0}, \boldsymbol{\omega} = \boldsymbol{\omega}_0. \tag{17}$$

If there is no spherical symmetry, but the ensuing perturbation of the interaction with other bodies may be neglected, then we have instead of (17)

$$\langle \boldsymbol{\delta\omega}, \frac{d}{dt}(\boldsymbol{J\omega}) \rangle = 0 \forall \boldsymbol{\delta\omega} \Rightarrow \frac{d}{dt}(\boldsymbol{J\omega}) = \boldsymbol{J\dot{\omega}} + \boldsymbol{\omega} \wedge (\boldsymbol{J\omega}) = \boldsymbol{0}. \tag{18}$$

This is a very complex equation for $\boldsymbol{\omega}$ since $\boldsymbol{J}$ depends on the orientation of the principal axes of inertia. The tensor $\boldsymbol{J}$ is obtained from its value in the initial orientation of the body by the transformation

$$\boldsymbol{J} = \boldsymbol{R}\boldsymbol{J}_0\boldsymbol{R}^T. \tag{19}$$

If we now transform the angular velocity vector from a vector in the inertial system to a vector in a coordinate system, rigidly connected to the body, $\boldsymbol{\omega}^* = \boldsymbol{R}^T\boldsymbol{\omega}$, then we obtain from equation (18) the famous Euler equations:

$$\boldsymbol{R}\boldsymbol{J}_0\boldsymbol{R}^T \frac{d}{dt}(\boldsymbol{R\omega}^*) + \frac{d}{dt}(\boldsymbol{R}\boldsymbol{J}_0\boldsymbol{R}^T)\boldsymbol{R\omega}^* = \boldsymbol{R}\left\{\boldsymbol{J}_0\dot{\boldsymbol{\omega}}^* + \boldsymbol{\omega}^* \wedge (\boldsymbol{J}_0\boldsymbol{\omega}^*)\right\} = \boldsymbol{0}$$

or

$$\boldsymbol{J}_0\dot{\boldsymbol{\omega}}^* + \boldsymbol{\omega}^* \wedge (\boldsymbol{J}_0\boldsymbol{\omega}^*) = \boldsymbol{0}, \dot{\boldsymbol{J}}_0 = \boldsymbol{0}. \tag{20}$$

The earth has no perfect spherical symmetry. The third principal moment of inertia is not exactly equal to the other two principal moments, but is slightly larger. If we take the third principal axis of inertia as one of the coordinate axes, rigidly connected to the earth, then we have the following equations for the components of the vector of angular velocity $\boldsymbol{\omega}$:

$$J_1\dot{\omega}_1^* + (J_3 - J_1)\omega_2^*\omega_3^* = 0,$$
$$J_1\dot{\omega}_2^* + (J_1 - J_3)\omega_1^*\omega_3^* = 0. \qquad (21)$$
$$J_3\dot{\omega}_3 = 0.$$

In addition to $\omega_3^* =$ constant these equations permit a periodic solution for ω_1^* and ω_2^* with a frequency of $^{(J_3-J_1)}\!/_{J_1}\,\omega_3^*$. This wobble of the earth is actually very well observed, giving rise to what is called the variation of latitude. The amplitude of the motion is but small.

The motion of the celestial bodies in our solar system can already with great accuracy be simulated by computation, neglecting the deviations from spherical symmetry. For our solar system the theory for n celestial bodies is then summarized by

$$\mathrm{E} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left\{ (\tfrac{1}{2}\,\mathrm{m_i}\dot{\boldsymbol{r}}_i \circ \dot{\boldsymbol{r}}_i) + (\tfrac{1}{2}\,\mathrm{J_i}\boldsymbol{\omega}_\mathrm{i} \circ \boldsymbol{\omega}_\mathrm{i}) \right\} - \sum_{\mathrm{i}\rangle\mathrm{j}}^{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{n}} \frac{\mathrm{G m_i m_j}}{\left\{ (\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{j}) \circ (\boldsymbol{r}_\mathrm{i} - \boldsymbol{r}_\mathrm{j}) \right\}^{1/2}} = \text{constant}. \qquad (22)$$

5
Motion and deformation of chemically stable, material bodies

In (22) the partial derivatives of the so-called gravitational potential with respect to the position vectors lead to the forces by which the bodies interact in their motion. We observe that bodies can also strongly mutually influence their motion by surface contact. A theory, that supplies contact forces, is acquired in an indirect way, since the contact forces depend on the material properties. While the gravitational forces may be given in terms of a vectorfield of forces per unit mass, $\boldsymbol{f}$, the contact forces are defined as **stress vectors** per unit surface area, $\boldsymbol{t}$. Considering one body we write analogously to (5)

$$\int_{\partial\mathrm{B}} \langle \boldsymbol{t}, \delta\dot{\boldsymbol{r}} \rangle \mathrm{dA} + \int_{\mathrm{B}} \langle \rho(\boldsymbol{f} - \ddot{\boldsymbol{r}}), \delta\dot{\boldsymbol{r}} \rangle \mathrm{dV} = 0 \forall \delta\dot{\boldsymbol{r}}. \qquad (23)$$

For the description of the motion of rigid bodies we use (11) and (13):

$$\int_{\partial\mathrm{B}} \langle \boldsymbol{t}, \delta\dot{\boldsymbol{r}}^c + \delta\,\boldsymbol{\omega} \wedge \boldsymbol{r}^* \rangle \mathrm{dA} +$$
$$\int_{\mathrm{B}} \langle \boldsymbol{\rho} \left\{ \boldsymbol{f} - \ddot{\boldsymbol{r}}^c - \dot{\boldsymbol{\omega}} \wedge \boldsymbol{r}^* - \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{r}^*) \right\}, \delta\dot{\boldsymbol{r}}^c + \delta\,\boldsymbol{\omega} \wedge \boldsymbol{r}^* \rangle = 0 \;\forall \delta\dot{\boldsymbol{r}}^c, \delta\,\boldsymbol{\omega},$$

or

$$\boldsymbol{F} = \mathrm{m}\ddot{\boldsymbol{r}}^c, \qquad \boldsymbol{M} = \frac{\mathrm{d}}{\mathrm{dt}}(\boldsymbol{J}\boldsymbol{\omega}),$$

$$\boldsymbol{F} = \int_{\partial\mathrm{B}} \boldsymbol{t}\mathrm{dA} + \int_{\mathrm{B}} \rho\boldsymbol{f}\mathrm{dV}, \qquad \boldsymbol{M} = \int_{\partial\mathrm{B}} \boldsymbol{r}^* \wedge \boldsymbol{t}\mathrm{dA} + \int_{\mathrm{B}} \rho\boldsymbol{r}^* \wedge \boldsymbol{f}\mathrm{dV}. \qquad (24)$$

The inertia tensor $\boldsymbol{J}$ depends according to (19) on the orthogonal transformation $\boldsymbol{R}$, for which we have the differential equation of the first order

$$\dot{\boldsymbol{R}} = \boldsymbol{\Omega}\boldsymbol{R},$$

in addition to the first order differential equation for $\boldsymbol{\omega}$. When the external force $\boldsymbol{F}$ and the external *moment* $\boldsymbol{M}$ are supplied as functions of time the motion of the body can be simulated by integration of the **equations of motion** starting at a time, for which the initial conditions are known. Through the non-linearity in the equations for the components of the moment there arise a number of remarkable phenomena, like gyroscopic effects and the wobble of the earth, that was touched upon in the previous paragraph.

For *deformable bodies* in the first place conservation of mass must be ensured. The position vector $\boldsymbol{r}$ of a so-called **material point** of the body is to be considered as a time dependent vectorfield, subjected only to conditions of continuity and piecewize differentiability as long as the cohesion of the material of the body is being maintained. Then the **law of conservation of mass** is expressed by

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathrm{B}}\rho\mathrm{dV}=\int_{\mathrm{B}}(\dot{\rho}+\rho\mathrm{div}\dot{\boldsymbol{r}})\mathrm{dV}=0,\ \text{or}\ \dot{\rho}+\rho\mathrm{div}\dot{\boldsymbol{r}}=0. \tag{25}$$

Though the value of E in (22) being constant in time, as a basis for the theory of motion for the celestial bodies in our solar system, could be designated as the law of conservation of energy, for the existence of a contribution to the gravitational potential there always must be two bodies. Such a contribution does not have an identifiable carrier, this in contrast to the contributions to the so-called **kinetic energy.** The contributions to the kinetic energy are expressed in terms of the velocities of the individual bodies. In the equation of motion (5), usually set forth as *Newton's law,*

$$\langle \mathrm{m}\ddot{\boldsymbol{r}}-\boldsymbol{f},\dot{\boldsymbol{r}}\rangle=0\forall\dot{\boldsymbol{r}}\Rightarrow \mathrm{m}\ddot{\boldsymbol{r}}=\boldsymbol{f},$$

the gravitational potential is hidden behind the force vector $\boldsymbol{f}$.

The scalar product $\langle \boldsymbol{f},\dot{\boldsymbol{r}}\rangle$ is designated as the work per unit of time, or as the **power** supplied by the external force $\boldsymbol{f}$. In the case of undeformable bodies the power supplied to a body is fully stored as kinetic energy of this body. Also deformable bodies are subjected to the law of conservation of energy, which is then designated as the **first law of thermodynamics.** However for deformable bodies the energy stored by the body consists of the sum of the kinetic energy and the **internal energy**.

With this internal energy as a function of the momentary state of the material of the body we define a model for this particular material. This implies that the first law of thermodynamics has physical content only in connection with a model of the momentary state and with the availability of equations, that determine changes of this state with time.

In the continuum theory for the internal energy we introduce a scalar field e, that has the physical dimension of energy per unit mass (m^2s^{-2}). The local value of e has a physical significance only if it is multiplied with a small, but finite mass $\rho\Delta\mathrm{V}$. Since the internal energy should also represent the kinetic energy of the temperature motion of the atoms, it is only defined for a conglomeration of a very large number of atoms. The more or less violent vibrations of the atoms about their equilibrium positions are at contact experienced by the observer as hot or cold. As a measure for hot and cold the notion of **temperature** is introduced. Temperature introduces a fourth physical dimension. In addition to length, measured in meters, mass, measured in kilograms, and time intervals, measured in seconds, we now have temperature, to be measured in degrees. The energy exchange without a net resultant force through the contact surface between bodies is represented by the product of the **absolute temperature** ϑ, non-negative by definition, and an **entropy velocity vector** $\dot{\boldsymbol{h}}$. Like the contact stress vector the contact temperature depends on the material properties and hence can only be defined in an indirect way. The temperature scale in degrees is derived from the properties of a so-called perfect gas, to be discussed further on. In the continuum theory for the motion of any body the law of conservation of energy is formulated as follows:

$$\int_{\partial B}\left\{\langle \boldsymbol{t},\dot{\boldsymbol{r}}\rangle-\langle \boldsymbol{n},\vartheta\dot{\boldsymbol{h}}\rangle\right\}dA+\int_{B}\left\{\langle \rho(\boldsymbol{f}-\ddot{\boldsymbol{r}}),\dot{\boldsymbol{r}}\rangle-\rho\dot{e}\right\}dV=0. \qquad (26)$$

Here $\boldsymbol{n}$ is the outward directed normal in the point of the surface under consideration. The velocities, $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{h}}$, may be arbitrarily chosen initial velocities of the process, to be designated as $\delta\dot{\boldsymbol{r}}$ and $\delta\dot{\boldsymbol{h}}$.

Following **Gibbs** we consider the internal energy function to be a **function of state**, of which the value is, but for a constant, determined by the value of the so-called **state variables** of the model. Changes of state are **reversible** in the so-called **state space**. In state space the variable time is absent, because the state variables by definition define a state of equilibrium. The difference between values of state variables at two consecutive points of time is of course a consequence of a time dependent process, but this process can not be described in the state space.

If the spatial gradients of the state variables are not too large, the small but finite mass element for which the local value of the internal energy as determined by the local values of the state variables is still physically meaningful, can be thought of as being **embedded** in the **physical space.** In this physical space, in which changes are generally **irreversible,** the rates of change of the state variables must be related to the mass velocities and the entropy velocities. We attribute to each material point of a body an internal energy as if we were dealing with a state of equilibrium, in which this internal energy is determined by the local values of the state variables of the model under consideration. We speak of the principle of **local equilibrium** [2]. In the admissibility of this description of the changes, that take place in the material of the body, lies the strength and the limitation of the continuum model of the material.

6
Navier-Stokes model for a gas

In the continuum theory the model of a gas is usually formulated in terms of an internal energy, that depends solely on the temperature, i.e. on the temperature motions of the gas molecules. This corresponds with a function e , that only depends on the local value of the mass density ρ and on the local value of the entropy per unit mass, s :

$$e=e(\rho,s). \qquad (27)$$

Just as the mass density this local entropy may vary with the divergence of the velocity field, in this case of the entropy velocity field. But in contrast with the change in mass density in (25) as a consequence of the divergence of the mass velocity field, by the **second law of thermodynamics** the change in entropy is never smaller than the influx of entropy. In terms of the so-called **Clausius-Duhem inequality,** with the aid of a non-negative entropy production term σ, the continuum formulation of the second law of thermodynamics reads as follows:

$$\rho\dot{s}+div\dot{\boldsymbol{h}}=\rho\sigma\,,\ \sigma\geq 0\,. \qquad (28)$$

For the mathematical description of the deformation we consider the square of the length of a line element in a material point of the continuum, determined by the inner product of the infinitesimal vector $d\boldsymbol{r}$ The expression for the time derivative of this inner product comprises a symmetric tensor:

$$dl^2=d\boldsymbol{r}\circ d\boldsymbol{r},\quad \frac{d}{dt}(dl^2)=d\boldsymbol{r}\circ\left\{(grad\dot{\boldsymbol{r}}+grad^T\dot{\boldsymbol{r}})d\boldsymbol{r}\right\}.$$

The tensor

$$\boldsymbol{D} = \tfrac{1}{2}(\,grad\dot{\boldsymbol{r}} + grad^{T}\dot{\boldsymbol{r}}\,) \tag{29}$$

is denoted as the **rate of deformation tensor**. In case of rigid translations and rotations of a body the length of any material line element is invariant and this tensor equals the zero tensor in all material points of the body. The local rate of change of volume, $div\dot{\boldsymbol{r}} = tr\boldsymbol{D}$, can be split off. Then we obtain the so-called **deviator** of the tensor $\boldsymbol{D}$,

$$\boldsymbol{D}^{d} = \boldsymbol{D} - \tfrac{1}{3} div\dot{\boldsymbol{r}}\boldsymbol{I}\,. \tag{30}$$

In order to transform the surface integral in (26) into a volume integral we introduce **Cauchy's theorem**, expressing the stress vector $\boldsymbol{t}$ at the surface as a contraction of a **stress tensor** $\boldsymbol{T}$, defined in the interior of the body, with the outward normal $\boldsymbol{n}$ at the surface point under consideration:

$$\boldsymbol{t} = \boldsymbol{nT}, \boldsymbol{T} = \boldsymbol{T}^{d} + \tfrac{1}{3} tr\boldsymbol{T}\boldsymbol{I}. \tag{31}$$

In the Navier-Stokes model of a gas the stress tensor $\boldsymbol{T}$ has a spherically symmetric component, determined by a state variable p, while the remaining components are a consequence of an irreversible process of exchange of **momentum**, $\rho\dot{\boldsymbol{r}}$, in the gas. Neighbouring layers in the continuum model exchange in the real gas molecules with different velocities without a net mass transport. This part of the stress tensor is denoted by $\boldsymbol{T}^{ir}$.

Now the equation (26) for conservation of energy can be written as one volume integral by means of the *divergence theorem*. If we substitute the expressions for $\dot{\rho}$ and $\dot{s}$ from (25) and (28) , we obtain

$$\begin{aligned}
&\int_{\partial B}\left\{\langle \boldsymbol{nT},\dot{\boldsymbol{r}}\rangle - \langle \boldsymbol{n}\vartheta,\dot{\boldsymbol{h}}\rangle\right\}dA + \int_{B}\left\{\langle \rho(\boldsymbol{f}-\ddot{\boldsymbol{r}}),\dot{\boldsymbol{r}}\rangle - \rho\left(\frac{\partial e}{\partial \rho}\dot{\rho} + \frac{\partial e}{\partial s}\dot{s}\right)\right\}dV = \\
&\int_{B}\Big[\langle\{\rho(\boldsymbol{f}-\ddot{\boldsymbol{r}}) + div\boldsymbol{T}\},\dot{\boldsymbol{r}}\rangle + \left(\rho^{2}\frac{\partial e}{\partial \rho} - p\right)div\dot{\boldsymbol{r}} + \left(\frac{\partial e}{\partial s} - \vartheta\right)div\dot{\boldsymbol{h}} - \frac{\partial e}{\partial s}\rho\sigma + \\
&\tfrac{1}{3} tr\boldsymbol{T}^{ir} div\dot{\boldsymbol{r}} + \langle \boldsymbol{T}^{ird}, \boldsymbol{D}^{d}\rangle - \langle grad\vartheta,\dot{\boldsymbol{h}}\rangle + \tfrac{1}{2}\langle(\boldsymbol{T}^{ir} - \boldsymbol{T}^{irT}), grad\dot{\boldsymbol{r}}\rangle\Big]dV = 0.
\end{aligned} \tag{32}$$

For equation (32) to be satisfied for arbitray $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{h}}$ the following equations must hold

$$\begin{aligned}
&\rho(\boldsymbol{f}-\ddot{\boldsymbol{r}}) + div\boldsymbol{T} = \boldsymbol{0}, \\
&p = \rho^{2}\frac{\partial e}{\partial \rho} \quad , \quad \vartheta = \frac{\partial e}{\partial s} \quad , \quad \boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}^{ir}, \quad \boldsymbol{T}^{ir} = \boldsymbol{T}^{irT}, \\
&\vartheta\rho\sigma = \tfrac{1}{3} tr\boldsymbol{T}^{ir} div\dot{\boldsymbol{r}} + \langle \boldsymbol{T}^{ird}, \boldsymbol{D}^{d}\rangle - \langle grad\vartheta,\dot{\boldsymbol{h}}\rangle.
\end{aligned} \tag{33}$$

The simplest positive definite expression for the **energy dissipation** $\vartheta\rho\sigma$, that satisfies (33) for arbitrary $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{h}}$, is given by

$$\vartheta\rho\sigma = \tfrac{1}{3}(2\mu + 3\lambda)(div\dot{\boldsymbol{r}})^{2} + 2\mu\boldsymbol{D}^{d}\circ\boldsymbol{D}^{d} + \frac{\vartheta}{k}\dot{\boldsymbol{h}}\circ\dot{\boldsymbol{h}}. \tag{34}$$

Here we have used the customary **constants of viscosity** μ and λ**,** as well as the **heat conduction coefficient** k. Often it is assumed that $2\mu + 3\lambda = 0$ gives the best description of the phenomenon of viscosity in a gas.

If we require that the expressions (33) and (34) are identically equal to each other, then the Navier-Stokes model is complemented to a mathematical description, that enables us to simulate all kinds of phenomena of gas flow:

$$\tfrac{1}{3} tr\boldsymbol{T}^{ir} = \tfrac{1}{3}(2\mu + 3\lambda) div\dot{\boldsymbol{r}}, \boldsymbol{T}^{ird} = 2\mu\boldsymbol{D}^{d}, \tag{35}$$

$$\vartheta\dot{\boldsymbol{h}} = -k\,grad\vartheta\,. \tag{36}$$

The expression for the internal energy, corresponding to the gas law

$$p = R\rho\vartheta \ , \tag{37}$$

formulated with an initial mass density ρ_0, an initial temperature ϑ_0, and an initial entropy s =0, is given by

$$e = c_v \vartheta_0 \left\{ \left(\rho / \rho_0 \right)^{\gamma} exp\ s / c_v - 1 \right\}, \gamma = R / c_v \ . \tag{38}$$

As material constants appear in this expression the **specific heat at constant volume**, c_v, and the so-called **gas constant** R, characteristic for the gas under consideration. Though the gas law (37) defines a family of so-called **perfect gases**, one for each value of R, any gas at low enough mass density approaches a perfect gas with a particular value of R. Hence the temperature scale is defined in terms of a gasthermometer with a gas appropriate for the temperature range to be considered (for instance helium with $R = 2.08.10^3 m^2 s^{-2\circ} K^{-1}$ for lower temperatures and nitrogen with $R = 2.97.10^2 m^2 s^{-2\circ} K^{-1}$ for higher temperatures). The absolute temperature is expressed in terms of degrees **Kelvin**, based upon the fixed temperature of the triple point of H_2O. The triple point is the temperature at which ice, water, and water vapor are all in equilibrium with one another. In the definition of the Kelvin scale the temperature assigned to the triple point is $273.16^{\circ}\mathrm{K}$ exactly, corresponding on the Celcius scale to $0.01^{\circ}\mathrm{C}$. The Celcius scale is based upon a temperature difference of 100 degrees between the steam point of water, $100^{\circ}\mathrm{C}$, and the ice point, $0^{\circ}\mathrm{C}$. On the Kelvin scale, defined by

$$\vartheta = 273.16 \frac{(p \setminus \rho)}{(p \setminus \rho)_{\text{triple}}}{}^{\circ}\mathrm{K},$$

this temperature difference between the ice point, $273.1502^{\circ}\mathrm{K}$, and the steam point, $373.146^{\circ}\mathrm{K}$, is very nearly preserved.

7
Model for the deformation of solids with a natural reference state

In contrast to the memory-free Navier-Stokes model for a gas, a model for the deformation of a solid must have a built-in memory in the form of so-called **elasticity**. Perfect elasticity may be defined as the capacity of the material under consideration to fully restore the initial geometry, if the values of the state variables that appear in the internal energy function are brought back to their initial values. A description, in which the material in state space can be characterized by an invariant function for the internal energy, takes the existence of a so-called **natural reference state** for granted. Then it is assumed, that for a finite, be it possibly very small neighbourhood of a material point of the continuum model, the geometrical configuration of the atoms in relation to each other differs from the configuration in the natural reference state by a linear transformation of line elements in that point:

$$\boldsymbol{dr} = \boldsymbol{F}^e d\boldsymbol{\xi}. \tag{39}$$

In the case of permanent perfectly elastic behaviour for each material point of the continuum model the position vector $\boldsymbol{\xi}$ and the initial position vector $\boldsymbol{r}_0$ are identical, $\boldsymbol{\xi} \equiv \boldsymbol{r}_0$. Then there exists an indestructible bond between the configuration of the atoms and the material points of the continuum. By the motion of the material points of the continuum not only the average mass motion is given, but for the elementary

particles of the elastic material holds, just as for the material points of the continuum, that neighbours remain neighbours.

However perfectly elastic behaviour generally may only be considered as a good approximation for very small deformations. While for the motion of gas molecules it is immediately clear, that the place of these molecules in the continuum model bears no relation to the configuration of the material points of the continuum, in a solid we need insight into the deformation process in order to see that also here neighbouring physical particles of the material generally do not remain neighbours. For instance in the case of crystalline materials (e.g. metals) slip processes caused by *dislocation movements* are responsible for the fact that continuously the individual atoms come under the influence of different atoms in the crystal lattice. However the internal energy of the continuum depends on changes in the mutual configuration of the physical particles with respect to the configuration in the natural reference state, independent of the fact whether this natural reference state in the material point under consideration is made up from the same particles as in the initial state. By (39) we express an experiment of thought, in which the atoms by a local deformation are reversibly restored to the configuration, corresponding to the configuration belonging to the initial values of the state variables in the internal energy function.

The **elastic deformation tensor**, acting as state variable in the internal energy function, is defined as follows:

$$d\boldsymbol{r}\circ d\boldsymbol{r}-d\boldsymbol{\xi}\circ d\boldsymbol{\xi}=2d\boldsymbol{\xi}\circ(\boldsymbol{\eta}^{e}d\boldsymbol{\xi}),\ \boldsymbol{\eta}^{e}=\tfrac{1}{2}(\boldsymbol{F}^{eT}\boldsymbol{F}^{e}-\boldsymbol{I}). \tag{40}$$

The line element $d\boldsymbol{\xi}$, representing the local natural reference state in the continuum model, will for the material point under consideration generally change its length and orientation by inelastic deformation. The rate at which this happens can be characterized in the physical space by a tensor $\boldsymbol{\Lambda}^{p}$:

$$d\dot{\boldsymbol{\xi}}=\boldsymbol{\Lambda}^{p}d\boldsymbol{\xi}\Rightarrow d\dot{\boldsymbol{r}}=\dot{\boldsymbol{F}}^{e}\boldsymbol{F}^{e-1}d\boldsymbol{r}+\boldsymbol{F}^{e}\boldsymbol{\Lambda}^{p}\boldsymbol{F}^{e-1}d\boldsymbol{r}=(\boldsymbol{L}^{e}+\boldsymbol{L}^{p})d\boldsymbol{r}. \tag{41}$$

The so-called **velocity gradient tensor** $\boldsymbol{L}$ has been split into an elastic and an inelastic part. Also the deformation gradient tensor $\boldsymbol{D}$ and the **spin tensor** $\boldsymbol{\Omega}$ may be split into an elastic and an inelastic part:

$$\begin{aligned}&\boldsymbol{D}^{e}=\tfrac{1}{2}(\boldsymbol{L}^{e}+\boldsymbol{L}^{eT}),\ \boldsymbol{D}^{p}=\tfrac{1}{2}(\boldsymbol{L}^{p}+\boldsymbol{L}^{pT}),\\&\boldsymbol{\Omega}^{e}=\tfrac{1}{2}(\boldsymbol{L}^{e}-\boldsymbol{L}^{eT}),\ \boldsymbol{\Omega}^{p}=\tfrac{1}{2}(\boldsymbol{L}^{p}-\boldsymbol{L}^{pT}).\end{aligned} \tag{42}$$

We have

$$\boldsymbol{D}=\boldsymbol{D}^{e}+\boldsymbol{D}^{p},\ \dot{\boldsymbol{\eta}}^{e}=\boldsymbol{F}^{eT}\boldsymbol{D}^{e}\boldsymbol{F}^{e}. \tag{43}$$

Again by the divergence theorem the law of conservation of energy (26) is written as a condition for an integral over an arbitrary volume:

$$\begin{aligned}&\int_{\partial B}\left\{\langle\boldsymbol{n}\boldsymbol{T},\dot{\boldsymbol{r}}\rangle-\langle\boldsymbol{n}\vartheta,\dot{\boldsymbol{h}}\rangle\right\}dA+\int_{B}\left\{\langle\rho(\boldsymbol{f}-\ddot{\boldsymbol{r}}).\dot{\boldsymbol{r}}\rangle-\rho\left(\frac{\partial e}{\partial\boldsymbol{\eta}^{e}}\dot{\boldsymbol{\eta}}^{e}+\frac{\partial e}{\partial s}\dot{s}\right)\right\}dV=\\&\int_{B}\Big[\langle\left\{\rho(\boldsymbol{f}-\ddot{\boldsymbol{r}})+div\boldsymbol{T}\right\},\dot{\boldsymbol{r}}\rangle-\langle\left(\rho\frac{\partial e}{\partial\boldsymbol{\eta}^{e}}-\boldsymbol{F}^{e-1}\boldsymbol{T}\boldsymbol{F}^{eT-1}\right),\dot{\boldsymbol{\eta}}^{e}\rangle+\left(\frac{\partial e}{\partial s}-\vartheta\right)div\dot{\boldsymbol{h}}-\\&\frac{\partial e}{\partial s}\rho\sigma+\langle\boldsymbol{T},\boldsymbol{D}^{p}\rangle-\langle grad\vartheta,\dot{\boldsymbol{h}}\rangle\Big]dV=0.\end{aligned} \tag{44}$$

If we require this equation to be satisfied for arbitrary $\dot{\boldsymbol{r}}$ and $\dot{\boldsymbol{h}}$, then we obtain the equations of motion,

$$\rho(\boldsymbol{f}-\ddot{\boldsymbol{r}})+div\boldsymbol{T}=\boldsymbol{0}, \tag{45}$$

as well as the so-called **constitutive equations** for the stress tensor and for the temperature together with an expression for the energy dissipation:

$$\mathbf{F}^{\mathbf{e}-1}\mathbf{T}\mathbf{F}^{\mathbf{e}\mathbf{T}-1} = \rho\,\frac{\partial e}{\partial \boldsymbol{\eta}^e}, \qquad \vartheta = \frac{\partial e}{\partial s},$$

$$\vartheta\rho\sigma = \langle \boldsymbol{T}, \boldsymbol{D}^p \rangle - \langle grad\vartheta, \dot{\boldsymbol{h}} \rangle. \tag{46}$$

The equations for $\boldsymbol{D}^p$ and $\dot{\boldsymbol{h}}$ are, just as in the case of the Navier-Stokes model, obtained with the aid of an expression for the energy dissipation. In the first place we observe that the change of volume for a solid, that permits a thermodynamic description, according to experiments is a purely elastic phenomenon. As a consequence the dissipation term of the inelastic deformation is determined by the deviator of the stress tensor, because $tr\boldsymbol{D}^p = 0.$ Next we assume the inelastic rate of deformation to be determined by the energy dissipation as a function of the state variables, since without the introduction of new concepts there is no alternative. The inelastic rate of deformation tensor as the dual of the stress deviator can be defined, but for a scalar multiplying factor, by the derivative of the dissipation function with respect to the stress deviator. We put

$$\vartheta\rho\sigma = \langle \boldsymbol{T}^d, \boldsymbol{D}^p \rangle - \langle grad\vartheta, \dot{\boldsymbol{h}} \rangle \equiv \varphi(\vartheta, p, \boldsymbol{T}^d) + {}^{\vartheta}\!/_{k}\,\dot{\boldsymbol{h}} \circ \dot{\boldsymbol{h}},$$

$$\boldsymbol{D}^p = \lambda\frac{\partial \varphi}{\partial \boldsymbol{T}^d}, \qquad \lambda = \varphi\left(\langle \boldsymbol{T}^d, \frac{\partial \varphi}{\partial \boldsymbol{T}^d} \rangle\right)^{-1}, \quad \boldsymbol{\Omega}^p = \boldsymbol{0},$$

$$\vartheta\dot{\boldsymbol{h}} = -kgrad\vartheta. \tag{47}$$

For an isotropic material the internal energy function and the dissipation function contain the invariants of the tensors $\boldsymbol{\eta}^e$ and $\boldsymbol{T}^d$ as state variables. Together with the initial pressure and temperature, p_0 and ϑ_0, and with the customary material constants, the specific heat c_v, the coefficient of cubic thermal expansion α, the **bulk modulus** C, and the **shear modulus** G, the internal energy function *for small elastic deformations* has the following form:

$$\rho e = \rho\vartheta_0 s - p_0\,tr\boldsymbol{\eta}^e + \frac{\rho\vartheta_0}{2c_v}s^2 - \frac{C\alpha\vartheta_0}{c_v}s\,tr\boldsymbol{\eta}^e + {}^{1}\!/_{2}C\left(1 + \frac{C\alpha^2\vartheta_0}{\rho c_v}\right)\left(tr\boldsymbol{\eta}^e\right)^2 +$$

$$G tr\left(\boldsymbol{\eta}^{ed}\boldsymbol{\eta}^{ed}\right). \tag{48}$$

A wellknown, often applied isotropic dissipation function, that for $n \to \infty$ passes into the Von Mises yield criterion with a yield stress σ_y, has the following form:

$$\varphi = \gamma(\vartheta)\left(\frac{3tr(\boldsymbol{T}^d\boldsymbol{T}^d)}{2(\sigma_y)^2}\right)^n. \tag{49}$$

Here the function $\gamma(\vartheta)$ takes the temperature dependence of the energy dissipation into account.

The description of the material behaviour may be enhanced by the introduction of so-called hidden or **internal state variables**. In [1] it is shown how a description of anisotropic strainhardening of an initially isotropc material can be obtained.

In particular for large deformations, as well as for excessive strain gradients [3], one meets with the limitations of the continuum model. With the computing power, nowadays available, one tries to bridge the gap between insight into the slip processes on a microscale and the description in a continuum model. For crystalline materials

the deformation process is governed by the atomic bonds. There inelastic deformation is a result of boundary sliding and of dislocation movements and production, at higher temperatures together with diffusion. For macro-molecular materials the molecular entanglements play a predominant role. In what is now called **micro-mechanics** the problem has to be faced, that the concepts of stress tensor and of temperature from the continuum theory are virtually meaningless on the atomic and molecular scale. In so-called mesoscopic models one tries to mate the continuum concepts with the atomic and molecular models.

8
Model for Liquids

Liquids have properties, which are more or less in between the properties of gases and solids. We could characterize a liquid as a material without shape memory, but with a perfect volume elasticity. In view of the large stiffness against volume changes the internal energy function (48) with a shear modulus put equal to zero ($G = 0$) gives an appropriate description with $\boldsymbol{\eta}^e$ determined by

$$tr\dot{\boldsymbol{\eta}}^e = \left(\tfrac{2}{3} tr\boldsymbol{\eta}^e + 1\right) div\dot{\boldsymbol{r}} \approx div\dot{\boldsymbol{r}} \qquad (50$$

The pressure p being determined by the internal energy, the stress tensor has the same form as for a gas, $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}^{ir}$, and also the constitutive equations (35) and (36) do apply. Models for a more complicated behaviour are often denoted as non-Newtonian fluids.

9
Concluding remarks

The forgoing derivation of well known equations for models of rigid and deformable bodies has been presented for no other purpose than to show how a concise and consistent presentation of the theory of mechanics and continuum thermodynamics is possible, without the often undue emphasis laid upon the "laws of Newton" and without an unnecessarily elaborate introduction of the thermodynamic aspects. The ultimate test of a physical theory is agreement and (even more important) no contradictions between observed and simulated data for the object of the theory. The development of safe and reliable complex mechanical structures for use on land, at sea, in the air and in space, in the amazing technological endeavor of the twentieth century, has put the theory to test more than reasonably could be desired.

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